

## PRACTICE WITH SLOPE

- Want some basic practice first?  
[Introduction to the Slope of a Line](#)



([more mathematical cats](#)).

In a rush? Need to skip the concepts and jump right to the slope formula?  
(Please come back and explore the beautiful ideas at a later time.)

Consider the equation  $y = mx + b$ .

As we'll see in this section, every equation of this form graphs as a non-vertical line in the coordinate plane.

In this equation,  $m$  is the coefficient of the  $x$ -term, and  $b$  is a constant term. For example, in the equation  $y = 2x + 3$ , we have  $m = 2$  and  $b = 3$ .

The number  $m$  is called the **slope** of the line, and gives information about the 'slant' of the line. It answers questions like: "Is it an uphill or downhill line? How steep is it?"  
The purpose of this section is to explain why this is true.

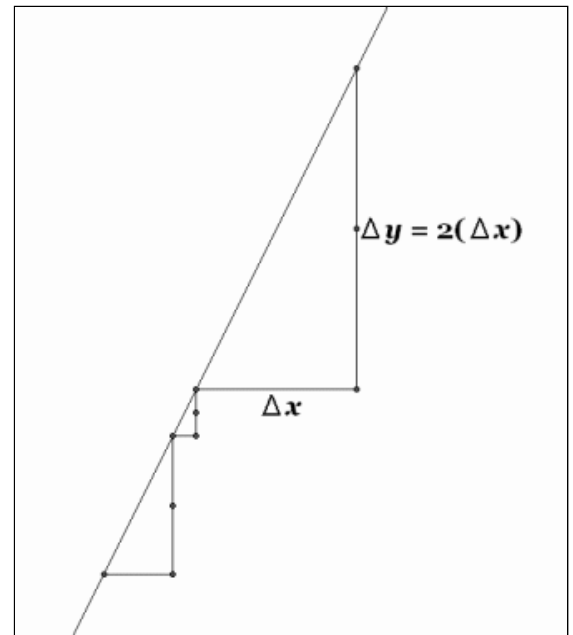
**WHY DOES  $y = mx + b$  GRAPH AS A LINE?**

**WHY DOES THE NUMBER  $m$  GIVE INFO ABOUT THE SLANT OF THIS LINE?**

The key idea (which we'll prove below) is that the equation  $y = mx + b$  defines a very special relationship between  $x$  and  $y$ : equal changes in  $x$  (the input) give rise to equal changes in  $y$  (the output).

Indeed, if  $x$  changes by an amount  $\Delta x$  (read aloud as 'delta ex' or 'change in ex'), then  $y$  changes by  $m\Delta x$ . That is,  $y$  changes  $m$  times as fast as  $x$ .  
This is a bit symbol-intensive, so let's look at a concrete example.

Suppose that  $m$  is 2. (See the sketch at right.)  
Then,  $y$  changes 2 times as fast as  $x$  (that is, twice as fast as  $x$ ).  
If (say)  $x$  changes by 1, then  $y$  will change by  $2(1) = 2$ .  
If (say)  $x$  changes by 5, then  $y$  will change by  $2(5) = 10$ , and so on.



Here's another way to look at it.

Imagine you're 'standing on' a point in the coordinate plane.

Let's force  $y$  to change twice as fast as  $x$ , and see what new points result:

If you take 1 step to the right (i.e., let  $x$  change by 1), then you'll have to take 2 steps up (i.e., let  $y$  change by 2).  
 Do it again—one step to the right, two steps up.  
 Do it again—one step to the right, two steps up.  
 (Repeat, if you want, with different changes in  $x$ .)  
 Hmm . . . what pattern are these points creating? A line!!

If we repeat the previous exercise with  $m = 100$ , think about the line that would result: one step to the right, 100 steps up. Pretty steep line!  
 So, clearly, the number  $m$  gives information about the steepness or ‘slant’ of the line.

The next part is a bit technical, so perhaps a break is in order first!

**PROOF THAT, IN THE EQUATION  $y = mx + b$ ,  
 EQUAL CHANGES IN  $x$  GIVE RISE TO EQUAL CHANGES IN  $y$**

Let  $(x_1, y_1)$  be a point on the graph of  $y = mx + b$ , so that  $y_1 = mx_1 + b$ .  
 That is, when you substitute  $x_1$  for  $x$  and  $y_1$  for  $y$ , the resulting equation is true.

Let  $x_1$  change by an amount  $\Delta x$ , to get a new  $x$ -value:  $x_2 = x_1 + \Delta x$

Let  $y_2$  represent the  $y$ -value corresponding to  $x_2$ .  
 Then,

$$\begin{aligned}
 y_2 &= mx_2 + b && \text{(find the } y\text{-value corresponding to } x_2 \text{; call it } y_2 \text{)} \\
 &= m(x_1 + \Delta x) + b && \text{(substitute } x_1 + \Delta x \text{ for } x_2 \text{)} \\
 &= mx_1 + m\Delta x + b && \text{(multiply out; i.e., use the Distributive Law)} \\
 &= (mx_1 + b) + m\Delta x && \text{(re-order and re-group)} \\
 &= y_1 + m\Delta x && \text{(look back: } mx_1 + b \text{ is precisely } y_1 \text{)}
 \end{aligned}$$

Thus,  $y$  has changed by  $m\Delta x$ .

Make sure you see this!

The starting point  $(x_1, y_1)$   
 is:

The ending point is:  $(x_1 + \Delta x, y_1 + m\Delta x)$

So, when  $x$  changes by  $\Delta x$ ,  $y$  changes by  $m\Delta x$ .

**A FORMULA FOR SLOPE**

Let's summarize things from the previous proof.

We started at  $(x_1, y_1)$ .

We ended at  $(x_2, y_2)$ .

The change in  $x$  is:  $\Delta x = x_2 - x_1$

A ‘change in  $x$ ’ is informally called a **run**, since it is in the left/right direction.

Memory device: Run to the right! Run to the left!

The change in  $y$  is:  $\Delta y = y_2 - y_1$

A 'change in  $y$ ' is informally called a *rise*, since it is in the up/down direction.

Memory device: Rise up!

And, we learned that the change in  $y$  is  $m\Delta x$ : in equation form,  $\Delta y = m\Delta x$ .

Solving this last equation for the slope,  $m$ , gives us an important formula:

### SLOPE OF A LINE

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be two different points on a non-vertical line.

Then, the slope of this line is:

$$m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

### ANALYSIS OF THE SLOPE FORMULA

There are lots of important things you should know about the slope formula:

- HOW TO USE THE SLOPE FORMULA:

To find the slope of a non-vertical line, you can choose any two different points on the line.

Take the difference of the  $y$ -values. This is your 'rise'.

Take the difference of the  $x$ -values. This is your 'run'.

Divide rise by run, and you've got your slope!

- BE CAREFUL—ORDER IS IMPORTANT!

Once you do your first subtraction, the next subtraction must be done *in the same order*.

For example, suppose your two points are  $(1, 5)$  and  $(3, 12)$ .

If you say  $\Delta x = 3 - 1 = 2$ , then you must say  $\Delta y = 12 - 5 = 7$ , and  $m = \frac{7}{2}$ .

Or:

If you say  $\Delta x = 1 - 3 = -2$ , then you must say  $\Delta y = 5 - 12 = -7$ , and  $m = \frac{-7}{-2} = \frac{7}{2}$ .

Either way is fine.

But DON'T use (say)  $3 - 1$  for one and  $5 - 12$  for the other!

- RUN:

If the change in  $x$  is  $2$ , you go to the *right* two.

If the change in  $x$  is  $-2$ , you go to the *left* two.

'Run' is positive to the right, and negative to the left.

- RISE:

If the change in  $y$  is  $2$ , you go *up* two.

If the change in  $y$  is  $-2$ , you go *down* two.

'Rise' is positive UP, and negative DOWN.

- HORIZONTAL LINES:

Since two points on a horizontal line have the same  $y$ -value,

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0.$$

The slope of a horizontal line is zero.

- **VERTICAL LINES:**

Since two points on a vertical line have the same  $x$ -value, an attempt to compute the slope fails:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0} = \text{Can't continue!}$$

Division by zero is not allowed.

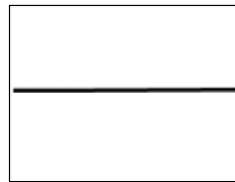
Thus, you can't compute the slope of a vertical line.

We say that vertical lines have NO slope.

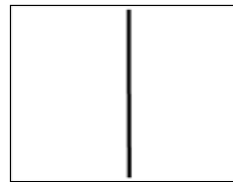
- **ZERO SLOPE versus NO SLOPE:**

Note that 'zero slope' and 'no slope' mean two entirely different things!

Horizontal lines have zero slope; vertical lines have no slope.



horizontal line,  
zero slope



vertical line,  
no slope

- **UPHILL LINES, POSITIVE SLOPE:**

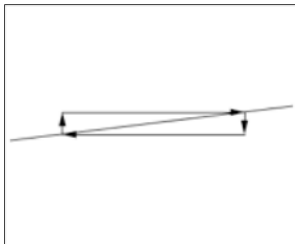
Imagine you're standing on a line in the coordinate plane, walking from left to right. If you're going uphill, then we'll informally call this an 'uphill line'.

On uphill lines, if you rise-and-run from point-to-point, one of two situations occur:

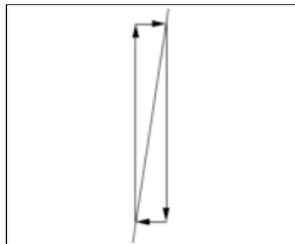
- rise and run are both positive, so slope =  $\frac{+}{+}$  = positive
- rise and run are both negative, so slope =  $\frac{-}{-}$  = positive

Thus, uphill lines have positive slopes.

**UPHILL LINES, POSITIVE SLOPES**

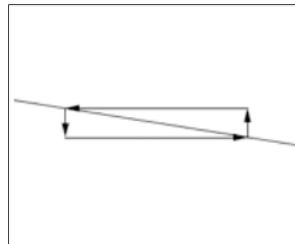


gradual uphill,  
small positive slope

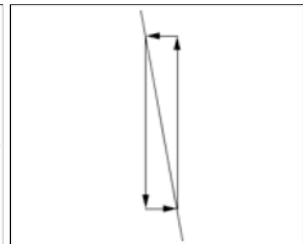


steep uphill,  
large positive slope

**DOWNHILL LINES, NEGATIVE SLOPES**



gradual downhill,  
small negative slope



steep downhill,  
large negative slope

- **DOWNHILL LINES, NEGATIVE SLOPE:**

Imagine you're standing on a line in the coordinate plane, walking from left to right. If you're going downhill, then we'll informally call this a 'downhill line'.

On downhill lines, if you rise-and-run from point-to-point, one of two situations occur:

- rise is positive and run is negative, so slope =  $\frac{+}{-}$  = negative
- rise is negative and run is positive, so slope =  $\frac{-}{+}$  = negative

Thus, downhill lines have negative slopes.

**EXAMPLES:****Question:**

Find the slope of the line through  $(-1, 3)$  and  $(2, -5)$ .

**Solution:**

$$\text{slope} = \frac{-5 - 3}{2 - (-1)} = \frac{-8}{3} = -\frac{8}{3}$$

The answers in this exercise are reported as fractions, in simplest form.

**Question:**

Find the slope of the line through  $(2, 5)$  and  $(-7, 5)$ .

**Solution:**

$$\text{slope} = \frac{5 - 5}{-7 - 2} = \frac{0}{-9} = 0$$

Recall that zero, divided by any nonzero number, is zero.  
This is a horizontal line.

**Question:**

Find the slope of the line through  $(2, 5)$  and  $(2, -7)$ .

**Solution:**

$$\text{slope} = \frac{-7 - 5}{2 - 2} = \frac{-12}{0} = \text{can't continue}$$

Division by zero is not allowed.  
This line has no slope.  
It is a vertical line.

**Question:**

Horizontal lines have (circle one):    no slope        zero slope

**Solution:**

zero slope

**Question:**

Vertical lines have (circle one):    no slope        zero slope

**Solution:**

no slope

**Question:**

Suppose a fraction has a zero in the numerator, and a nonzero denominator.  
What is the value of the fraction?

**Solution:**

zero

**Question:**

Suppose a fraction has a zero in the denominator, and a nonzero numerator.  
What is the value of the fraction?

**Solution:**

it is not defined

**Question:**

Start at a point  $(x, y)$  on a line.  
To get to a new point, move up 3 and to the right 4.  
What is the slope of the line?

**Solution:**

Rise is 3; run is 4.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3}{4}$$

**Question:**

Start at a point  $(x, y)$  on a line.  
To get to a new point, move down 2 and to the left 6.  
What is the slope of the line?

**Solution:**

Rise is  $-2$ ; run is  $-6$ .

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-2}{-6} = \frac{1}{3}$$

**Question:**

Start at a point  $(x, y)$  on a line.  
To get to a new point, move straight up 5 units.  
What is the slope of the line?

**Solution:**

no slope (vertical line)

**Question:**

Start at a point  $(x, y)$  on a line.  
To get to a new point, move directly to the left 5 units.  
What is the slope of the line?

**Solution:**

zero slope (horizontal line)

**Question:**

Suppose you are walking along a line, moving from left to right.  
You are going uphill.  
Then, the slope of the line is (circle one):    positive        negative

**Solution:**

positive